AN EXPONENTIAL MODEL FOR THE SPECTRUM OF A SCALAR TIME SERIES

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Technical Report 12, Series 2

Department of Statistics

PRINCETON UNIVERSITY

April 1972

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Summary

A new class of parametric models for the spectrum of a scalar time series is proposed, in which the logarithm of the spectral density function is represented by a finite Fourier series. Two alternative parameter estimation procedures are described, and the use of a fitted model to provide forecasts of future values is discussed. The model has been compared with the more conventional autoregressive/moving-average model, and the results of their comparison are given.

This research was supported by the Office Of Naval Research on contract N00014-67-A-0151-0017 awarded to the Department of Statistics, Princeton University

1. Introduction

Although there has been much discussion of the fitting of parametric models to time series, essentially only one such model, or class of models, has been considered. This is the auto-regressive/moving-average model, which is discussed by Box and Jenkins (1970, Chapter 3), for instance. In this model, it is assumed that the observed series $\{X_t\}$ is related to an unobserved white noise series $\{\varepsilon_t\}$ by

$$X_t = \sum_{r=1}^{p} \phi_r X_{t-r} + \varepsilon_t - \sum_{r=1}^{q} \theta_r \varepsilon_{t-r} \quad . \tag{1}$$

The spectral density function of the series is then

$$g(\omega) = \frac{\tau^2}{2\pi} \left| \frac{1 - \sum_{r=1}^{q} \theta_r e^{ir\omega}}{1 - \sum_{r=1}^{p} \phi_r e^{ir\omega}} \right|^2$$

where τ^2 is the variance of ε_t.

The model discussed in this paper was motivated by the observation that the logarithm of an estimated spectral density function is often found to be a fairly well-behaved function, and could thus be approximated by a truncated Fourier series. The form which has been chosen for this model is

$$g(\omega) = \frac{\tau^2}{2\pi} \exp\left(2 \sum_{r=1}^{p} \theta_r \cos r\omega\right) \quad . \tag{2}$$

The evenness of g restricts the exponent to be a cosine series, which is written in this form for later convenience.

It is, of course, possible to approximate any well-behaved function by a truncated Fourier series. However, the model (2) would be of use only if p were small. This raises the question of whether such spectra arise in practice. An answer to this question will evidently be found only in extensive application of the model. The model (1), used with small values of p and q, has been remarkably successful at fitting practical data. In many cases, a similarly good fit could be expected with the model (2).

In section 2 we discuss maximum likelihood estimation of the parameters of the model, and the asymptotic distribution of the estimates, while in section 3 we consider some inefficient estimates. The construction of predictions of future values of the series is described in section 4. The model was fitted to a number of time series discussed by Box and Jenkins (1970, p. 575), and the results are contained in section 5. Finally, in section 6 we examine briefly the effect of one type of departure from our assumptions.

2. Maximum Likelihood Parameter Estimation

Walker (1964) discusses the estimation of a general model of the form

$$g(\omega) = \frac{\tau^2}{2\pi} h(\omega;\underset{\sim}{\theta}),$$

where

$$h(\omega;\underset{\sim}{\theta}) = \left|1 + \sum_{r=1}^{\infty} \alpha_r(\underset{\sim}{\theta}) e^{ir\omega}\right|^2.$$

Now this condition is equivalent to

$$\int_{-\pi}^{\pi} \log\{h(\omega;\underset{\sim}{\theta})\}d\omega = 0,$$

for all values of $\underset{\sim}{\theta}$, and hence the model (2) falls into this class.

Walker shows that, if the series is Gaussian, then the log-likelihood of the parameters, given observed data $x_1,\ldots,x_n$, is approximately

$$-n \log(2\pi) - \frac{n}{2}\log \tau^2 - \frac{n}{2\tau^2}\int_{-\pi}^{\pi} \frac{I(\omega)}{h(\omega;\underset{\sim}{\theta})}\, d\omega ,$$

where

$$I(\omega) = (2\pi n)^{-1}\left|\sum_{r=1}^{n} x_t e^{it\omega}\right|^2 ,$$

the periodogram of $x_1,\ldots,x_n$. Thus $\underset{\sim}{\theta}$ may be estimated by minimising

$$\int_{-\pi}^{\pi} \frac{I(\omega)}{h(\omega;\underset{\sim}{\theta})}\, d\omega , \qquad (3)$$

and τ^2 may be estimated by the minimised value.

Walker also shows the resulting estimates, $\hat{\underset{\sim}{\theta}}$ and $\hat{\tau}^2$, are consistent, and furthermore that the distribution of $\sqrt{n}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}_0)$ converges to $N\{\underset{\sim}{0}, \underset{\sim}{W}(\underset{\sim}{\theta}_0)^{-1}\}$ where

$$\underset{\sim}{W}(\underset{\sim}{\theta}) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\{\frac{\partial}{\partial\underset{\sim}{\theta}} \log h(\omega;\underset{\sim}{\theta})\}\{\frac{\partial}{\partial\underset{\sim}{\theta}} \log h(\omega;\underset{\sim}{\theta})\}' d\omega .$$

For the model (2), it is easily verified that $\underset{\sim}{W}(\underset{\sim}{\theta})$ is identically equal to $\underset{\sim}{I}$, which is one reason for the use of $2\cos r\omega$ in that model.

With other models, direct minimisation of (3) is not the usual procedure for obtaining the estimates $\hat{\underset{\sim}{\theta}}$ and $\hat{\tau}^2$. However, for the model (2), the result that $\underset{\sim}{W}(\underset{\sim}{\theta}) = \underset{\sim}{I}$ makes such a procedure simple. For a Newton-Raphson minimisation gives rise to an iterative scheme in which

$$\hat{\underset{\sim}{\theta}}_{n+1} = \hat{\underset{\sim}{\theta}}_n - \underset{\sim}{V}(\hat{\underset{\sim}{\theta}}_n)^{-1}\underset{\sim}{v}(\hat{\theta}_n) ,$$

where $\underset{\sim}{v}(\hat{\underset{\sim}{\theta}}_n)$ and $\underset{\sim}{V}(\hat{\underset{\sim}{\theta}}_n)$ are the vector and matrix, respectively, of first and second derivatives of (3), evaluated in each case at $\hat{\underset{\sim}{\theta}}_n$. Now $\underset{\sim}{V}(\hat{\underset{\sim}{\theta}}_n)$ can be approximated by $\underset{\sim}{V}(\underset{\sim}{\theta}_0)$, which can further be approximated by its expectation, since its variance is readily seen to be $O(n^{-1})$. But $E\{\underset{\sim}{V}(\underset{\sim}{\theta}_0)\}$ may itself be approximated by $2\tau^2\underset{\sim}{W}(\underset{\sim}{\theta}_0) = 2\tau^2\underset{\sim}{I}$. Replacing τ^2 by its n'th iterated estimate, $\hat{\tau}^2_n$, which is of course just (3) evaluated at $\hat{\theta}_n$, we obtain a modified iterative solution, in which

$$\hat{\underset{\sim}{\theta}}_{n+1} = \hat{\underset{\sim}{\theta}}_n - \frac{1}{2\tau_n^2}\underset{\sim}{v}(\hat{\theta}_n) .$$

Note that

$$\frac{1}{2\hat{\tau}_n^2} v(\hat{\theta}_n) = \frac{1}{2\hat{\tau}_n^2} \int_{-\pi}^{\pi} \frac{I(\omega)}{h(\omega;\theta_n)} \frac{\partial}{\partial \underset{\sim}{\theta}} \log\{h(\omega;\theta)\}\Big|_{\hat{\theta}_n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2\pi I(\omega)}{\hat{\tau}_n^2 h(\omega;\hat{\theta}_n)} \underset{\sim}{c}(\omega) d\omega \quad ,$$

where $\underset{\sim}{c}(\omega)' = (\cos \omega, \ldots, \cos p\omega)$. Now the first factor in the integrand is just $I(\omega)/\hat{g}_n(\omega)$, where $\hat{g}_n(\omega)$ is the n'th iterated estimate of the spectral density function $g(\omega)$. Thus the vector of modifications contains the Fourier coefficients of the normalised periodogram, and in particular these vanish at the minimising values $\hat{\underset{\sim}{\theta}}$.

3. Inefficient Parameter Estimates

Some consistent but inefficient estimates of the parameters of our model are suggested by the following observation. Let $\omega_j = 2\pi j/n$, and $m = [(n-1)/2]$. Then the distribution of the periodogram ordinates $I(\omega_1),\ldots,I(\omega_m)$, may be approximated by that of m independent exponential random variables with means

$$g(\omega_j) = \frac{\tau^2}{2\pi} \exp(2 \sum_{r=1}^{p} \theta_r \cos r\omega_j) .$$

Suppose that $Z_1,\ldots,Z_m$ have precisely this distribution, and let $\zeta_j = \log Z_j$, $j = 1,\ldots,m$. Then the ζ_j are uncorrelated, and

$$E(\zeta_j) = \psi(1) + \log\frac{\tau^2}{2\pi} + 2 \sum_{r=1}^{p} \theta_r \cos r\omega_j ,$$

and

$$\operatorname{var}(\zeta_j) = \psi'(1) .$$

Here ψ is the di-gamma function, whence $\psi(1)$ is $-\gamma$, and γ is Euler's constant, 0.57722. Also $\psi'(1)$ is $\pi^2/6 = 1.64493$.

Let

$$t_r^{(1)} = \frac{2}{n} \sum_{j=1}^{m} \zeta_j \cos r\omega_j , \quad r = 1,\ldots,p ,$$

and

$$u^{(1)} = 2\pi \exp(-\psi'(1) + \frac{2}{n} \sum_{j=1}^{n} \zeta_j) .$$

Then it is easily verified that

$$E(t_r^{(1)}) = \theta_r + O(n^{-1}) ,$$

$$\operatorname{var}(t_r^{(1)}) = \frac{1}{n}\psi'(1) + O(n^{-1}),$$

$$\operatorname{cov}(t_r^{(1)},\theta_s^{(1)}) = O(n^{-1}),$$

and indeed that the distribution of $\sqrt{n}(\underset{\sim}{t}^{(1)} - \underset{\sim}{\theta})$ converges to $N(\underset{\sim}{0},\psi'(1)\underset{\sim}{I})$.

Now if we define $\underset{\sim}{\theta}^{(1)}$ in terms of $\log I(\omega_j)$, $j = 1,\ldots m$, in the same way that $t^{(1)}$ is defined from ζ_j , $j = 1,\ldots,m$, then clearly under suitable conditions the same asymptotic distribution will be valid. Comparison with the results for the approximate maximum likelihood procedure described in the previous section shows that the estimates $\underset{\sim}{\theta}^{(1)}$ have efficiency $\psi'(1)^{-1}$, that is, around 61%.

Jones (1964) has discussed the calculation of quantities very similar to $\underset{\sim}{\theta}^{(1)}$, but recommends that the periodogram be smoothed somewhat before taking logarithms. Even in terms of $Z_1,\ldots,Z_m$, this gives rise to rather difficult problems in determining the properties of the estimates. However, a similar procedure would be as follows. Suppose that m has a fairly small factor k , and $m = km_1$. Then the Z's may be smoothed in a sense, by dividing them into m_1 adjacent blocks of k , and summing the blocks. Let

$$Z_j^{(k)} = \sum_{\ell=(j-1)k+1}^{jk} Z_\ell \ , \ j = 1,\ldots,m \ .$$

and $\zeta_j^{(k)} = \log Z_j^{(k)}$. Clearly the expectations of Z's in the same block will be similar. Suppose they were identical, i.e.

$$\sum_{r=1}^{p} \theta_r \cos r\omega_\ell = \sum_{r=1}^{p} \theta_r \cos r\omega_{jk} \ , \ \ell = (j-1)k+1,\ldots,jk$$
$$j = 1,\ldots m_1 \ .$$

It would then be true that

$$E(\zeta_j^{(k)}) = \psi(k) + 2\sum_{r=1}^{p}\theta_r \cos r\omega_{jk} + \log\frac{\tau^2}{2\pi} ,$$

$$\mathrm{var}(\zeta_j^{(k)}) = \psi'(k) ,$$

$j = 1,\ldots m_1$. Thus one may define new statistics

$$t_r^{(k)} = \frac{2k}{n}\sum_{j=1}^{m_1}\zeta_j^{(k)}\cos r\omega_j , \quad r = 1,\ldots p,$$

$$u^{(1)} = 2\pi \exp(-\psi'(k) + \frac{2k}{n}\sum_{j=1}^{m_1}\zeta_j^{(k)}) .$$

The asymptotic distribution of these statistics as $n \to \infty$ for fixed k is such that the distribution of $\sqrt{n}(\underset{\sim}{t}^{(k)}-\underset{\sim}{\theta})$ converges to $N(\underset{\sim}{0},k\psi'(k)\underset{\sim}{I})$. As before, we define $\underset{\sim}{\theta}^{(k)}$ in terms of logarithms of sums of periodogram values, in the same way that $\underset{\sim}{t}^{(k)}$ is defined from the Z's. Then under suitable conditions, the distribution of $\sqrt{n}(\underset{\sim}{\theta}^{(k)}--\underset{\sim}{\theta})$ will converge to the normal distribution. In particular, the efficiency of $\underset{\sim}{\theta}^{(k)}$ relative to the approximate maximum likelihood solution discussed in the previous section is $\{k\psi'(k)\}^{-1}$, values of which are given in Table 1.

Clearly we may also define estimates of τ^2, to be defined in terms of $I(\omega_j)$ analogously with $u^{(k)}$, $k = 1,2,\ldots$. The first of these was discussed by Davis and Jones (1968) and more recently by Cleveland (1971). The efficiencies of these will, for Gaussian data, follow the same progression as in Table 1.

For non-Gaussian data, the situation is not so clear. For instance, the variance of an efficient estimate of τ^2, as derived by Walker (1964) for a linear process, depends on the value of the fourth cumulant of the innovation process. Thus the argument we have used should not be expected to be valid in that case.

4. Construction of Predictions

For the model (1), predictions of future values may be constructed directly from this equation, as in Box and Jenkins (1970, Chapter 5). Since the parameters in the exponential model (2) have no similar direct interpretation, the construction of predictions is based on more general theory; see for instance Hannan (1970, Chapter 3). For a series $\{X_t\}$ with spectral density function $g(\cdot)$, the central problem is to find a function $\alpha(\cdot)$ of a complex variable z, satisfying

(i) $g(\omega) = \frac{\tau^2}{2\pi}|\alpha(e^{i\omega})|^2$ for some τ^2,

(ii) $\alpha(0) = 1$

(iii) α has neither zeroes nor poles in the unit disc $|z| < 1$.

For if the Taylor series expansion of $\alpha(\cdot)^{-1}$ is

$$\alpha(z)^{-1} = 1 - \sum_{s=1}^{\infty} \beta_s z^s ,$$

then the best one-step predictor of X_t is

$$\hat{X}_t = \sum_{s=1}^{\infty} \beta_s X_{t-s} ,$$

and its mean squared error, $E\{(X_t - \hat{X}_t)^2\}$, is τ^2.

Predictions may be made further into the future recursively. For the best two-step predictor of X_t is found by predicting X_{t-1} by the above formula with t replaced by $t-1$, and then substituting the predicted value into the original formula. An n-step prediction is found by repeating this operation. The mean

squared errors of these successive predictions are found from the Taylor series expansion of $\alpha(\cdot)$. For if

$$\alpha(z) = 1 + \sum_{s=1}^{\infty} \alpha_s z^s ,$$

then the mean squared error of the n-step prediction is

$$\tau^2(1 + \sum_{s=1}^{n-1} \alpha_s^2) , \quad n > 1 .$$

For the effect on these mean squared errors of using fitted parameter values, see Bloomfield (1972).

General numerical methods are available for obtaining the coefficients $\{\alpha_s\}$ and $\{\beta_s\}$. However, for the exponential model (2) we may use the structure of the model to obtain these coefficients analytically. For

$$\alpha(z) = \exp(\sum_{r=1}^{p} \theta_r z^r)$$

is clearly the required function. Suppose first that $p = 1$, so that $\alpha(z) = \exp(\theta_1 z)$. Then clearly $\alpha_s = \theta_1^s/s!$ and $\beta_s = (-1)^{s-1}\alpha_s$. Next suppose that $p > 1$. Then

$$\alpha(z) = \prod_{r=1}^{p} \exp(\theta_r z^r) .$$

The Taylor series of each factor of this product is easily written down, and the Taylor series of the product is just the convolution. The coefficients $\{\beta_r\}$ may be obtained similarly.

The auto-covariances of a series $\{X_t\}$ may be obtained by a similar argument. As before, we suppose first that $p = 1$. Then the spectral density function is

$$\frac{\tau^2}{2\pi} \exp\{\theta_1(e^{i\omega} + e^{-i\omega}) = \frac{\tau^2}{2\pi} \sum_{s=-\infty}^{\infty} I_s(2\theta_1)(e^{i\omega})^s ,$$

where $I_s(\cdot)$ is a modified Bessel function (Abramowitz and Stegun, 1964, p. 374). Hence the auto-covariance of lag s is $\tau^2 I_s(2\theta_1)$. When $p > 1$, the spectral density function is a product of similar terms, and thus the autocovariances are convolutions of sequences of Bessel functions. Notice, however, that the convolutions in this case involve infinite summations, whereas those involved in obtaining the coefficients $\{\alpha_s\}$ and $\{\beta_s\}$ involved only finite sums.

If one goes back over the data which were used to fit the model, one may use the fitted predictor to predict each value from the preceding value, and hence find the error of this prediction. Had one used the true predictor to do this, the errors would have been uncorrelated with each other. Now it is easily shown that the periodogram of these errors may be found approximately, up to a constant multiple, by dividing the periodogram of the data by the estimated spectral density function. However, it was shown in section 2 that if the efficient fitting procedure described there is used, then the first p Fourier coefficients of this normalised periodogram vanish. This implies, of course, that the sample autocovariances of the fitted prediction errors vanish, for lags 1 to p, to this order of approximation. An examination of the fitting procedure of section 2 shows that no other model with the same number of parameters can achieve this. Thus of all models containing a given number of parameters, the exponential model (2) gives the whitest residuals, in this sense.

By contrast, when a p-parameter autoregressive model has been fitted, the residuals are uncorrelated with the previous p values of the original series. Thus in an autoregressive model, one detects lack of fit by finding autocorrelation amongst the residuals. One must evidently look elsewhere in the case of fitting the exponential model.

5. Results of Fitting the Model

In order to compare the exponential model with the established autoregressive-moving average models, it was fitted to six series studied by Box and Jenkins (1970, pp. 525 ff). For several of these series, Box and Jenkins first differenced the data in order to render it more tractable. In each case, the same degree of differencing was used, and then the same number of parameters were fitted. The results are presented in Table 2.

The exponential model was fitted by the efficient iterative scheme described in section 2. Initial values were taken to be the first of the inefficient estimates of section 3. The criterion for convergence was based on the sum of squares of the modifications to the parameters multiplied by the series length and divided by the numer of parameters being fitted. The iteration was terminated when this quantity first fell below 0.01, or in other words, when the root mean square modification first fell below one tenth of the common asymptotic standard deviation. The multi-parameter models were fitted sequentially, each new parameter requiring around four iterations.

The goodness of fit of the models has been assessed by the residual variance. Since this estimates the one-step prediction error variance which would be achieved by using the fitted model, it provides useful criterion with a practical interpretation. For three of the series, the exponential model provides virtually as good a fit as the respective best autoregressive-moving average model. For another two, however, the fit is considerably worse.

For those series, series A and E , the addition of two more parameters to the exponential model did little to improve the fit.

While these results contain no evidence that time series exist which fit the exponential model better than the autoregressive-moving average model, at least they show that the two models provide comparable fits for certain series. Since there are p+1 distinct autoregressive-moving average models, each containing p parameters, one must expect the exponential model to be rather less flexible. Further experience will be required to determine whether this lack of flexibility will be sufficient to out-weigh the computational and statistical simplicity of the exponential model.

6. Departure from Assumptions

When fitting one of a sequence of parametric models to a time-series, one is never more than trying to approximate the structure of the series, if indeed it possesses a stationary structure. Thus it will rarely be true that the spectrum of the series belongs to the class of spectra being fitted. Since this is one of the assumptions underlying the result of Walker (1964) which has been used in section 2, it is instructive to examine what happens if that assumption is dropped.

Suppose then that the spectrum of the series is $g_0(\omega)$, and that the class of models under consideration is of the form

$$\frac{\tau^2}{2\pi} \; h(\omega;\underset{\sim}{\theta}) \; ,$$

as in section 2. For parameter values $\underset{\sim}{\theta}$, the mean squared error of the corresponding one-step predictor is

$$\int_{-\pi}^{\pi} \{g_0(\omega)/h(\omega;\underset{\sim}{\theta})\}d\omega \, ,$$

We shall regard the closest member of the family as being defined by parameters $\underset{\sim}{\theta}_0$, τ_0^2, where $\underset{\sim}{\theta}_0$ minimises this integral and τ_0^2 is the minimum value. It may be shown that the estimates defined in section 2 are consistent estimates of these quantities. In that sense, that estimation procedure and the present definition of closeness are well-matched. However, the asymptotic distribution of the estimates is no longer the same. Whilst asymptotic joint normality with zero means still holds, the asymptotic dispersion

matrix is now $\underset{\sim}{V}(\underset{\sim}{\theta}_0)^{-1}\underset{\sim}{U}(\underset{\sim}{\theta}_0)\underset{\sim}{V}(\underset{\sim}{\theta}_0)^{-1}$, where

$$\underset{\sim}{V}(\underset{\sim}{\theta}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} g_0(\omega)\left\{\frac{2\underset{\sim}{h}_1(\omega,\underset{\sim}{\theta})\underset{\sim}{h}_1(\omega,\underset{\sim}{\theta})'}{h(\omega,\underset{\sim}{\theta})^3} - \frac{\underset{\sim}{H}_2(\omega,\underset{\sim}{\theta})}{h(\omega\underset{\sim}{\theta})^2}\right\} d\omega \quad ,$$

$$\underset{\sim}{U}(\underset{\sim}{\theta}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{g_0(\omega)}{h(\omega;\underset{\sim}{\theta})^4}\,\underset{\sim}{h}_1(\omega;\underset{\sim}{\theta})\underset{\sim}{h}_1(\omega;\underset{\sim}{\theta})'d\omega \quad ,$$

and $\underset{\sim}{h}_1$ and $\underset{\sim}{H}_2$ denote the vector and matrix of first and second θ-derivatives of h , respectively.

For the p-parameter exponential model (2), it is easily seen that

$$\underset{\sim}{h}_1(\omega;\underset{\sim}{\theta}) = 2h(\omega;\underset{\sim}{\theta})\underset{\sim}{c}(\omega) \quad ,$$

$$\underset{\sim}{H}_2(\omega;\underset{\sim}{\theta}) = 4h(\omega;\underset{\sim}{\theta})\underset{\sim}{c}(\omega)\underset{\sim}{c}(\omega)' \quad ,$$

where $\underset{\sim}{c}(\omega)' = (\cos\omega,\ldots,\cos p\omega)$. Thus in this case,

$$\underset{\sim}{V}(\underset{\sim}{\theta}) = \pi^{-1}\int_{-\pi}^{\pi} \frac{g_0(\omega)}{h(\omega;\underset{\sim}{\theta})}\,\underset{\sim}{c}(\omega)\underset{\sim}{c}(\omega)'\,d\omega \quad ,$$

$$\underset{\sim}{U}(\underset{\sim}{\theta}) = \pi^{-1}\int_{-\pi}^{\pi}\left\{\frac{g_0(\omega)}{h(\omega;\underset{\sim}{\theta})}\right\}^2\underset{\sim}{c}(\omega)\underset{\sim}{c}(\omega)'\,d\omega \quad ,$$

Then it is easily verified that

$$\underset{\sim}{U}(\underset{\sim}{\theta}) - \underset{\sim}{V}(\underset{\sim}{\theta})^2 = \pi^{-1}\int_{-\pi}^{\pi}\left\{\frac{g_0(\omega)}{h(\omega;\underset{\sim}{\theta})}\underset{\sim}{I} - \underset{\sim}{V}(\underset{\sim}{\theta})\right\}\underset{\sim}{c}(\omega)\underset{\sim}{c}(\omega)'\left\{\frac{g_0(\omega)}{h(\omega;\underset{\sim}{\theta})}\underset{\sim}{I} - \underset{\sim}{V}(\underset{\sim}{\theta})\right\}d\omega \quad ,,$$

since

$$\int_{-\pi}^{\pi} \underset{\sim}{c}(\omega)\underset{\sim}{c}(\omega)'d\omega = \pi\underset{\sim}{I} \quad .$$

Since the integrand is symmetric and nonnegative definite, it follows that $\underset{\sim}{U}(\theta) - \underset{\sim}{V}(\theta)^2$ and hence also $\underset{\sim}{V}(\theta)^{-1}\underset{\sim}{U}(\theta)\underset{\sim}{V}(\theta)^{-1} - \underset{\sim}{I}$ are nonnegative definite.

Thus the asymptotic dispersion matrix for $\hat{\underset{\sim}{\theta}}$ derived in section 2 is in reality a lower bound, only achieved when the spectrum of the series being analysed belongs to the class of spectra being fitted. It seems plausible that the same should hold for an arbitrary model, but we have been unable to extend the result beyond the exponential model.

It follows that when fitting the exponential model as an approximation, misleadingly significant fitted values will arise more frequently than would otherwise be expected, when the true parameter value is, in fact, zero. Extra caution will be needed to avoid the inclusion of unnecessary parameters.

Table 1. Variance and Efficiency of $\underset{\sim}{t}^{(k)}$

k	n var $t_r^{(k)} = k\psi'(k)$	eff. $\underset{\sim}{t}^{(k)}$ (%)	$\psi'(k)$
1	1.64493	61	1.64493
2	1.28986	78	0.64493
3	1.18479	84	0.39493
4	1.13528	88	0.28382
5	1.10660	90	0.22132
6	1.08792	92	0.18132
7	1.07478	93	0.15354
8	1.06504	94	0.13313

Table 2. Results of fitting the model

Series	Degree of differencing	Number of parameters	Residual Variance	
			Box and Jenkins	Exponential
A(1)	0	2	0.097(2)	0.146
A	1	1	0.101	0.164
B	1	1	52.2	52.2
C	1	1	0.018	0.023
D	0	1	0.090	0.121
D	1	1	0.096	0.096
E	0	2	228	297
E	0	3	218	295
F	0	2	113	115

(1) The labelling is as in Box and Jenkins (1970, p. 524).

(2) These values are taken from Table 7.13 of Box and Jenkins (1970, p. 239).

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